# Some Banach space characterizations of the $\Delta_2$ condition.

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**Abstract**

In [1] J. Alexopoulos has shown that if $\Phi \in \Delta_2$ and if its complement $\Psi$ satisfies $\lim_{t\to\infty} \frac{\Psi(ct)}{\Psi(t)} = \infty$ for some $c > 0$ then a bounded set $K \subset L_\Phi$ is relatively weakly compact if and only if $K$ has equi-absolutely continuous norms. Even though all such $\Phi$ fail the $\nabla_2$ condition we do not know whether the theorem is applicable to all $\Phi \in \Delta_2 \setminus \nabla_2$. In this paper we make some progress towards a generalization of this theorem. In particular we show that an $N$-function $\Phi \notin \nabla_2$ if and only if every weakly null sequence $(c_n \chi_{E_n})$ in $E^\Phi(\mu)$ has equi-absolutely continuous norms. Other characterizations of the $\Delta_2$ condition are given.

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## 1 Introduction and background

Throughout this discussion $(\Omega, \Sigma, \mu)$ is a non-atomic, separable probability space. We begin by summarizing the necessary facts from the theory of Orlicz spaces. For a detailed account, the reader could consult chapters one and two in [5] or [7].

**Definition 1.1** *A function $\Phi$ is an $N$-function if and only if $\Phi$ is continuous, even and convex satisfying*

1. $\lim_{x\to 0} \frac{\Phi(x)}{x} = 0$;
2. $\lim_{x\to\infty} \frac{\Phi(x)}{x} = \infty$;
3. $\Phi(x) > 0$ *if* $x > 0$.

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**Definition 1.2** *For an N-function* $\Phi$ *define* $\Psi(x) = \sup\{t|x| - \Phi(t) : t \geq 0\}$. *Then* $\Psi$ *is an N-function and it is called the complement of* $\Phi$.

Observe that $\Phi$ is the complement of its complement $\Psi$.

**Definition 1.3** *Two N-functions* $\Phi$ *and* $F$ *are called equivalent if there are positive constants* $k_1$, $k_2$ *and* $x_0$ *so that*

$$F(k_2 x) \leq \Phi(x) \leq F(k_2 x) \text{ for all } x \geq x_0.$$

Given an $N$-function $\Phi$, the corresponding space of $\Phi$-integrable functions is defined as follows:

**Definition 1.4** *For an N-function* $\Phi$ *and a measurable* $f$ *define*

$$\mathbf{\Phi}(f) = \int_\Omega \Phi(f) d\mu.$$

*If* $\Psi$ *denotes the complement of* $\Phi$ *let*

$$L^\Phi = \left\{ f \text{ measurable} : \left| \int_\Omega fg d\mu \right| < \infty, \ \forall g \text{ with } \mathbf{\Psi}(g) < \infty \right\}$$

*The collection* $L^\Phi$ *is then a linear space. For* $f \in L^\Phi$ *define*

$$\|f\|_\Phi = \sup \left\{ \left| \int_\Omega fg d\mu \right| : \mathbf{\Psi}(g) \leq 1 \right\}$$

*Then* $(L^\Phi, \|\cdot\|_\Phi)$ *is a Banach space, called an Orlicz space. Moreover, letting* $\|f\|_{(\Phi)} = \inf \left\{ k > 0 : \mathbf{\Phi}\left(\frac{f}{k}\right) \leq 1 \right\}$ *be the Minkowski functional associated with the convex set* $\{f \in L^\Phi : \mathbf{\Phi}(f) \leq 1\}$, *we have that* $\|\cdot\|_{(\Phi)}$ *is an equivalent norm on* $L^\Phi$, *called the Luxemburg norm. Indeed,* $\|f\|_{(\Phi)} \leq \|f\|_\Phi \leq 2\|f\|_{(\Phi)}$, *for all* $f \in L^\Phi$.

**Theorem 1.5** *Let* $\Phi$ *be an N-function and let* $E^\Phi$ *be the closure of the bounded functions in* $L^\Phi$. *Then the conjugate space of* $(E^\Phi, \|\cdot\|_{(\Phi)})$ *is* $(L^\Psi, \|\cdot\|_\Psi)$, *where* $\Psi$ *is the complement of* $\Phi$.

**Definition 1.6** *An N-function* $\Phi$ *is said to satisfy the* $\Delta_2$ *condition* $(\Phi \in \Delta_2)$ *if* $\limsup_{x\to\infty} \frac{\Phi(2x)}{\Phi(x)} < \infty$. *That is, there is a* $K > 0$ *so that* $\Phi(2x) \leq K\Phi(x)$ *for large values of* $x$. *If the complement* $\Psi$ *of* $\Phi$ *satisfies the* $\Delta_2$ *condition then we say that* $\Phi$ *satisfies the* $\nabla_2$ *condition* $(\Phi \in \nabla_2)$.

**Definition 1.7** *We say that a collection* $\mathcal{K} \subset L^\Phi$ *has equi-absolutely continuous norms if and only if it is norm bounded and* $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ *so that* $\sup\{\|\chi_E f\|_\Phi : f \in \mathcal{K}\} < \varepsilon$ whenever $\mu(E) < \delta$.

**Theorem 1.8** *Let* $\Phi$ *be an N-function and* $\Psi$ *be its complement. Then the following statements are equivalent:*

1. $L^\Phi = E^\Phi$.

2. $L^{\Phi}=\{f \text{ measurable} : \mathbf{\Phi}(f)<\infty\}$.

3. *The dual of* $\left(L^{\Phi},\|\cdot\|_{(\Phi)}\right)$ *is* $\left(L^{\Psi},\|\cdot\|_{\Psi}\right)$.

4. $\forall\, f\in L^{\Phi}$, $\{f\}$ *has equi-absolutely continuous norms.*

5. $\Phi\in\Delta_2$.

In section 2 we develop criteria for an $N$-function $\Phi$ to satisfy the $\nabla_2$ condition. Specifically in theorem 2.3 we show that *An $N$-function $\Phi\notin\nabla_2$ if and only if every weakly null sequence $(c_n\chi_{E_n})$ in $L^{\Phi}(\mu)$ has equi-absolutely continuous norms.*

In section 3 we proceed to establish that if an $N$-function $\Phi$ does not satisfy the $\Delta_2$ condition then the space $E^{\Phi}$ is "$c_0$-rich" and as a result, there are weakly null sequences in $E^{\Phi}$ (even disjointly supported) that fail to have equi-absolutely continuous norms. In particular in corollary 3.4 we show that *if $\Phi\notin\Delta_2$ then there is a weakly null sequence $(f_n)\subset E^{\Phi}$ that fails to have equi-absolutely continuous norms.*

Finally in section 4 we close with some remarks relating relative weak compactness in Orlicz spaces to equi-absolute continuity of norms, and the presence of $c_0$ in $E^{\Phi}(\mu)$.

## 2 A characterization of the $\nabla_2$ condition

We begin with a lemma which takes advantage of the non-atomic nature of the measure to select disjoint sets of precise measures:

**Lemma 2.1** *Suppose that $\mathcal{K}\subset E^{\Phi}(\mu)$ does not have equi-absolutely continuous norms. Then $\exists\ \varepsilon_0>0$ so that for every positive null sequence $(a_n)$ there is a subsequence $(n_k)$ of the positive integers, a sequence $(f_k)\subset\mathcal{K}$ and a sequence $(A_k)\subset\Sigma$ of pairwise disjoint sets so that $\mu(A_k)=a_{n_k}$ and $\|f_k\chi_{A_k}\|\geq\varepsilon_0$.*

*Proof:* Since $\mathcal{K}$ does not have equi-absolutely continuous norms, $\exists\ \eta_0>0$ so that $\forall\ \delta>0$ there is $E_\delta\in\Sigma$ and $f\in\mathcal{K}$ so that $\|f\chi_{E_\delta}\|\geq\eta_0$. Let $\varepsilon_0=\frac{\eta_0}{2}$ and let $(a_n)$ be any positive null sequence.

Step 1: Let $n_1=1$ and choose $E_1\in\Sigma$ with $\mu(E_1)=a_{n_1}$ and $f_1\in\mathcal{K}$ with $\|f_1\chi_{E_1}\|\geq\eta_0$. Now $f_1$ has continuous norm and thus there is a $\delta_1>0$ so that if $\mu(E)<\delta_1$ then $\|f_1\chi_E\|<\frac{\eta_0}{2}$. Enlarge $E_1$ by adding a set of measure $\delta_1$ and let the new set be denoted by $E_1^+$. That is, $E_1\subset E_1^+$ and $\mu\left(E_1^+\right)=a_{n_1}+\delta_1$. Notice that if $A\subset E_1^+$ with $\mu(A)\geq a_{n_1}$ then $\mu\left(E_1^+\setminus A\right)\leq\delta_1$ and so

$$\frac{\eta_0}{2}>\left\|f_1\chi_{E_1^+\setminus A}\right\|\geq\|f_1\chi_{E_1}\|-\|f_1\chi_A\|\geq\eta_0-\|f_1\chi_A\|.$$

Hence $\|f_1\chi_A\|>\frac{\eta_0}{2}=\varepsilon_0$.

Step 2: Choose $n_2>n_1$ with $a_{n_2}<\frac{\delta_1}{4}$ and choose $E_2\in\Sigma$ with $\mu(E_2)=a_{n_2}$ and $f_2\in\mathcal{K}$ with $\|f_2\chi_{E_2}\|\geq\eta_0$. Now $f_2$ has continuous norm and thus there is a $0<\delta_2<\frac{\delta_1}{4}$ so that if $\mu(E)<\delta_2$ then $\|f_2\chi_E\|<\frac{\eta_0}{2}$.

Enlarge $E_2$ by adding a set of measure $\delta_2$ to obtain $E_2 \subset E_2^+$ and $\mu\left(E_2^+\right) = a_{n_2} + \delta_2$. Again, if $A \subset E_2^+$ with $\mu(A) \geq a_{n_2}$ then $\mu\left(E_2^+ \setminus A\right) \leq \delta_2$ and so

$$\frac{\eta_0}{2} > \left\| f_2 \chi_{E_2^+ \setminus A} \right\| \geq \| f_2 \chi_{E_2} \| - \| f_2 \chi_A \| \geq \eta_0 - \| f_2 \chi_A \| .$$

Hence $\| f_2 \chi_A \| > \frac{\eta_0}{2} = \varepsilon_0$.

Inductive step: Continue inductively and choose $n_{k+1} > n_k$ with $a_{n_{k+1}} < \frac{\delta_k}{4}$, $E_{k+1} \in \Sigma$ with $\mu(E_{k+1}) = a_{n_{k+1}}$ and $f_{k+1} \in \mathcal{K}$ with $\left\| f_{k+1} \chi_{E_{k+1}} \right\| \geq \eta_0$. Since $f_{k+1}$ has continuous norm, there is a $0 < \delta_{k+1} < \frac{\delta_k}{4}$ so that if $\mu(E) < \delta_{k+1}$ then $\| f_{k+1} \chi_E \| < \frac{\eta_0}{2}$. Enlarge $E_{k+1}$ by adding a set of measure $\delta_{k+1}$ to obtain $E_{k+1} \subset E_{k+1}^+$ and $\mu\left(E_{k+1}^+\right) = a_{n_{k+1}} + \delta_{k+1}$. Again, if $A \subset E_{k+1}^+$ with $\mu(A) \geq a_{n_{k+1}}$ then $\mu\left(E_{k+1}^+ \setminus A\right) \leq \delta_{k+1}$ and so

$$\frac{\eta_0}{2} > \left\| f_{k+1} \chi_{E_{k+1}^+ \setminus A} \right\| \geq \left\| f_{k+1} \chi_{E_{k+1}} \right\| - \| f_{k+1} \chi_A \| \geq \eta_0 - \| f_{k+1} \chi_A \| .$$

Hence $\| f_{k+1} \chi_A \| > \frac{\eta_0}{2} = \varepsilon_0$.

For each $k$, let $B_k = E_k^+ \setminus \bigcup_{j=k+1}^{\infty} E_j^+$. Then the $B_k$'s are pairwise disjoint. Furthermore

$$\begin{aligned}
\mu(B_k) &\geq \mu\left(E_k^+\right) - \sum_{j=k+1}^{\infty} \mu\left(E_j^+\right) \\
&= a_{n_k} + \delta_k - \sum_{j=k+1}^{\infty} \left(a_{n_j} + \delta_j\right) \\
&\geq a_{n_k} + \delta_k - \sum_{j=k}^{\infty} \frac{2\delta_k}{4^{j-k+1}} \\
&= a_{n_k} + \delta_k - 2\delta_k \sum_{j=1}^{\infty} \frac{1}{4^j} \\
&= a_{n_k} + \delta_k - \frac{2}{3}\delta_k \\
&> a_{n_k}.
\end{aligned}$$

Now choose any measurable $A_k \subset B_k$ with $\mu(A_k) = a_{n_k}$. Then $\| f_k \chi_{A_k} \| > \varepsilon_0$ and so the lemma is established. ∎

We next present a "Rosenthal's lemma" (see [3, pp.82]) type of result which can be found in [1].

**Lemma 2.2** *Let $X$ be a Banach space. Suppose that $(x_n) \subset X$ is weakly null and $(x_n^*) \subset X^*$ is weak$^*$ null. Then for each $\varepsilon > 0$ there is a subsequence $(n_k)$ of the positive integers, so that, for each positive integer $k$ we have*

$$\sum_{j \neq k} \left| \left\langle x_{n_j}^*, x_{n_k} \right\rangle \right| < \varepsilon.$$

*Proof:* Let $\varepsilon > 0$. Let $n_1 = 1$. Since $x_n^* \overset{\text{weak}^*}{\longrightarrow} 0$ there is an infinite subset $A_1$ of the positive integers so that $\sum_{j \in A_1} \left| \left\langle x_j^*, x_{n_1} \right\rangle \right| < \frac{\varepsilon}{2}$ . Since $x_n \to 0$ weakly and since $A_1$ is infinite, we can find $n_2 > n_1$ with $n_2 \in A_1$, so that $\left|\left\langle x_{n_1}^*, x_{n_2}\right\rangle\right| < \frac{\varepsilon}{2}$ . Similarly there is an infinite subset $A_2$ of $A_1$ so that $\sum_{j\in A_2}\left|\left\langle x_j^*, x_{n_2}\right\rangle\right| < \frac{\varepsilon}{2}$. Again choose $n_3 > n_2$ with $n_3 \in A_2$ so that $\left|\left\langle x_{n_1}^*, x_{n_3}\right\rangle\right| < \frac{\varepsilon}{4}$ and $\left|\left\langle x_{n_2}^*, x_{n_3}\right\rangle\right| < \frac{\varepsilon}{4}$ . There is an infinite subset $A_3$ of $A_2$ so that $\sum_{j\in A_3}\left|\left\langle x_j^*, x_{n_3}\right\rangle\right| < \frac{\varepsilon}{2}$. Choose $n_4 > n_3$ with $n_4 \in A_3$ so that $\left|\left\langle x_{n_i}^*, x_{n_4}\right\rangle\right| < \frac{\varepsilon}{6}$ for $i = 1,\ 2,\ 3$. Continue inductively to construct a sequence of infinite subsets of the positive integers, $A_1 \supset A_2 \cdots \supset A_k \supset \cdots$ and a sequence $n_1 < n_2 < \cdots$ of positive integers satisfying

1. $n_{k+1} \in A_k$ for all $k$.
2. $\sum_{j\in A_k}\left|\left\langle x_j^*, x_{n_{k+1}}\right\rangle\right| < \frac{\varepsilon}{2}$ for all $k$.
3. $\left|\left\langle x_j^*, x_{n_{k+1}}\right\rangle\right| < \frac{\varepsilon}{2k}$ for all $k$ and for $j = 1, 2, \ldots, k$.

Now for fixed positive integer $k$ we have

$$\begin{aligned}\sum_{j\neq k}\left|\left\langle x_{n_j}^*, x_{n_k}\right\rangle\right| &= \sum_{j=1}^{k-1}\left|\left\langle x_{n_j}^*, x_{n_k}\right\rangle\right| + \sum_{j=k+1}^{\infty}\left|\left\langle x_{n_j}^*, x_{n_k}\right\rangle\right| \\ &< \frac{\varepsilon}{2(k-1)}(k-1) + \sum_{j\in A_k}\left|\left\langle x_{n_j}^*, x_{n_k}\right\rangle\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

And so we are done. ∎

Now we are ready for our main result.

**Theorem 2.3** *An N-function $\Phi \notin \nabla_2$ if and only if every weakly null sequence $(c_n\chi_{E_n})$ in $L^\Phi(\mu)$ has equi-absolutely continuous norms.*

*Proof:* For the forward direction suppose that $\Phi \notin \nabla_2$ and assert that there is a weakly null sequence $(f_n) = (c_n\chi_{E_n}) \subset E^\Phi$ that fails to have equi-absolutely continuous norms. Use lemma 2.1 to find a sequence of pairwise disjoint measurable sets $(B_n)$ so that $(c_n\chi_{E_n\cap B_n})$ still fails to have equi-absolutely continuous norms. Since the complement $\Psi$ of $\Phi$ does not satisfy the $\Delta_2$ condition, there is a sequence of numbers $(s_n)$ with $s_n \nearrow \infty$ so that $\Psi(s_n) \geq 2^n\Psi\left(\frac{s_n}{2}\right)$ and $\frac{1}{\Psi(s_n)} \leq \mu(E_n\cap B_n)$ (if necessary, pass to a subsequence). Using lemma 2.1 again, there is an $\varepsilon_0 > 0$, a subsequence $(n_k)$ of the positive integers and a sequence $(A_k)$ of pairwise disjoint measurable sets so that $A_k \subset E_{n_k}\cap B_{n_k}$, $\|\chi_{A_k}f_{n_k}\|_\Phi > \varepsilon_0$ for all positive integers $k$ and $\mu(A_k) = \frac{1}{\Psi(s_{n_k})}$. For each $k$ let $g_k = s_{n_k}\chi_{A_k}$ and notice that

$$\left|\int_\Omega g_k f_{n_k}\,d\mu\right| = |c_{n_k}|\cdot s_{n_k}\cdot\mu(A_k) = |c_{n_k}|\cdot\mu(A_k)\cdot\Psi^{-1}\left(\frac{1}{\mu(A_k)}\right) = |c_{n_k}|\cdot\|\chi_{A_k}\|_\Phi = \|\chi_{A_k}f_{n_k}\|_\Phi > \varepsilon_0$$

Now let $g = \sum_{k=1}^\infty g_k$. Then

$$\int_\Omega \Psi\left(\frac{g}{2}\right)d\mu = \sum_{k=1}^\infty \Psi\left(\frac{s_{n_k}}{2}\right)\mu(A_k) \leq \sum_{k=1}^\infty \frac{1}{2^{n_k}}\Psi(s_{n_k})\frac{1}{\Psi(s_{n_k})} \leq 1$$

and so $g \in L^{\Psi}$ with $\|g\|_{(\Psi)} \leq 2$. Hence $g_k \in L^{\Psi}$ with $\|g_k\|_{(\Psi)} \leq 2$. For a fixed $f \in E^{\Phi}$, Hölder's Inequality yields

$$\left|\int f g_k d\mu\right| = \left|\int \chi_{A_k} f g_k d\mu\right| \leq \|\chi_{A_k} f\|_{\Phi} \cdot \|g_k\|_{(\Psi)} \leq 2\,\|\chi_{A_k} f\|_{\Phi}$$

But since $(A_k)$ are pairwise disjoint and $\mu$ is finite we have that $\mu(A_k) \to 0$ and as $f \in E^{\Phi}$, $f$ has absolutely continuous norm. Thus $\| \chi_{A_k} f \|_{\Phi} \to 0$. So $\int f g_k d\mu \to 0$. Hence $(g_k)$ is weak$^*$ null. By lemma 2.2 there is a subsequence $(k_j)$ of the positive integers so that for each $i$ we have $\sum_{j\neq i}\left|\int g_{k_j} f_{n_{k_i}} d\mu\right| < \frac{\varepsilon_0}{2}$.

Let $h = \sum_{j=1}^{\infty} g_{k_j}$.Then $\|h\|_{(\Psi)} \leq 2$ and so $h \in L^{\Psi}$. Since $(f_n)$ is weakly null, we must have $\int h f_{n_{k_m}} d\mu \to 0$ as $m \to \infty$. But for each positive integer $m$ we have

$$\begin{aligned}\left|\int_{\Omega} h f_{n_{k_m}} d\mu\right| &= \left|\int_{\Omega}\left(\sum_{j=1}^{\infty} g_{k_j}\right) f_{n_{k_m}} d\mu\right| \\ &\geq \left|\int_{\Omega} g_{n_{k_m}} f_{n_{k_m}} d\mu\right| - \sum_{j\neq m}\left|\int_{\Omega} g_{n_{k_j}} f_{n_{k_m}} d\mu\right| \\ &> \frac{\varepsilon_0}{2},\end{aligned}$$

which is a contradiction.

For the other direction suppose that $\Phi \in \nabla_2$. Since $(\Omega, \Sigma, \mu)$ is non-atomic, choose a sequence of measurable sets $(A_n)$ with $\mu(A_n) = \frac{1}{n}$. Then the sequence $\left(\Phi^{-1}(n)\chi_{A_n}\right) \subset E^{\Phi}$ is a weakly null sequence that does not have equi-absolutely continuous norms:

First notice that $\int_{\Omega} \Phi\left(\Phi^{-1}(n)\chi_{A_n}\right) d\mu = 1$ and so the Luxemburg norm $\left\|\Phi^{-1}(n)\chi_{A_n}\right\|_{(\Phi)} = 1$. Hence $\left(\Phi^{-1}(n)\chi_{A_n}\right)$ does not have equi-absolutely continuous norms. Now since $\Phi \in \nabla_2$ we have that the complement $\Psi$ of $\Phi$ satisfies the $\Delta_2$ condition and so if $f \in \left(E^{\Phi}\right)^* = L^{\Psi}$ we obtain

$$\begin{aligned}\left|\int_{\Omega} f \Phi^{-1}(n)\chi_{A_n} d\mu\right| &= \left|\int_{\Omega} f\chi_{A_n}\Phi^{-1}(n)\chi_{A_n} d\mu\right| \\ &\leq \|f\chi_{A_n}\|_{\Psi} \cdot \left\|\Phi^{-1}(n)\chi_{A_n}\right\|_{(\Phi)} \text{ (by Hölder's inequality)} \\ &= \|f\chi_{A_n}\|_{\Psi} \longrightarrow 0 \text{ as } n \to \infty \text{ since } f \text{ has continuous norm.}\end{aligned}$$

Hence $\left(\Phi^{-1}(n)\chi_{A_n}\right)$ is weakly null in $E^{\Phi}$. ∎

# 3 A characterization of the $\Delta_2$ condition

We first make an observation that is useful throughout this discourse:

**Proposition 3.1** *Every disjointly supported sequence in an Orlicz space is a monotonic basic sequence.*

*Proof:* If $(f_k)$ is such a sequence in $L^\Phi$ then for any sequence of scalars $(a_k)$ and any positive integers $m > n$ we have

$$\begin{aligned}\left\|\sum_{k=1}^n a_k f_k\right\|_\Phi &= \left\|\sum_{k=1}^n |a_k f_k|\right\|_\Phi \\ &\le \left\|\sum_{k=1}^m |a_k f_k|\right\|_\Phi \\ &= \left\|\sum_{k=1}^m a_k f_k\right\|_\Phi\end{aligned}$$

and so $(f_k)$ is as claimed. ∎

**Theorem 3.2** *Let $\Phi$ be an $N$–function and let $f \in L^\Phi$. Then $f \notin E^\Phi$ if and only if there is a sequence of disjoint measurable sets $(A_n)_{n=1}^\infty$ so that $(\chi_{A_n} f)_{n=1}^\infty = (f_n)_{n=1}^\infty$ is in $E^\Phi$ and $(f_n)$ is equivalent to the unit vector basis of $c_0$.*

*Proof:* Notice first that if $h \in L^\Phi$ with $\|h\|_\Phi > \varepsilon$ then there is a measurable set $A$ over which $h$ is bounded and such that $\|h\chi_A\|_\Phi > \varepsilon$. To see this let $h_n = h\chi_{[|h|\le n]}$. Then $|h_n| \nearrow |h|$ a.s. Hence if $g \in \tilde{L}^\Psi$ with $\Psi(g) \le 1$ and $\int_\Omega |gh|\, d\mu > \varepsilon$ we have that $|gh_n| \nearrow |gh|$ and so $\int_\Omega |gh_n|\, d\mu \nearrow \int_\Omega |gh|\, d\mu$. Thus for sufficiently large $n$ we have that $\|h_n\|_\Phi \ge \int_\Omega |gh_n|\, d\mu > \varepsilon$.

Now let $f \in L^\Phi \setminus E^\Phi$. Then $f$ does not have absolutely continuous norm and so we can find a sequence of disjoint measurable sets $(E_n)$ and an $\varepsilon_0 > 0$ so that for each $n$ we have $\|\chi_{E_n} f\|_\Phi > \varepsilon_0$. Now choose measurable subsets $A_n \subset E_n$ so that $f$ is bounded on $A_n$ and $\|\chi_{A_n} f\|_\Phi > \varepsilon_0$. Let $f_n = \chi_{A_n} f$. Then by proposition 3.1 $(f_n)$ is a monotonic basic sequence in $E^\Phi$ with $\inf_n \|f_n\|_\Phi \ge \varepsilon_0$. Furthermore for every $g \in L^\Psi$ we have

$$\begin{aligned}\sum_{n=1}^\infty \left|\int_\Omega g f_n\, d\mu\right| &\le \sum_{n=1}^\infty \int_\Omega |g f_n|\, d\mu \\ &= \int_\Omega |g| \sum_{n=1}^\infty |f_n|\, d\mu \\ &\le \int_\Omega |g|\cdot|f|\, d\mu \le \|g\|_{(\Psi)} \cdot \|f\|_\Phi < \infty\end{aligned}$$

Hence $\sum f_n$ is weakly unconditionally Cauchy. Hence by Bessaga–Pelczynski [2], $(f_n)$ is equivalent to the unit vector basis of $c_0$.

Conversely if $f \in L^\Phi$ and there is a sequence of disjoint measurable sets $(A_n)$ so that $(\chi_{A_n} f)$ is equivalent to $c_0$'s unit vector basis then $\inf_n \|\chi_{A_n} f\|_\Phi = \varepsilon_0 > 0$. Since the sets $A_n$ are disjoint, we have that $\mu(A_n) \to 0$ as $n \to \infty$ and so $f$ does not have absolutely continuous norm. Hence $f \notin E^\Phi$. ∎

We have the following immediate corollary:

**Corollary 3.3** *$\Phi \notin \Delta_2$ if and only if $E^\Phi$ contains a copy of $c_0$.*

*Proof:* If $\Phi \notin \Delta_2$ then there is a function $f \in L^\Phi \setminus E^\Phi$. Hence by theorem 3.2 there is a sequence of disjoint measurable sets $(A_n)$ so that $(\chi_{A_n} f) = (f_n) \subset E^\Phi$ and $(f_n)$ is equivalent to the unit vector basis of $c_0$. On the other hand, if $\Phi \in \Delta_2$ then $L^\Phi = E^\Phi$ contains no copies of $c_0$ (see [8]). ∎

**Corollary 3.4** *If $\Phi \notin \Delta_2$ then there is a weakly null sequence $(f_n) \subset E^\Phi$ that fails to have equi-absolutely continuous norms.*

*Proof:* This is immediate from theorem 3.2. ∎

As applications of corollary 3.3, other Banach space characterizations of $\Delta_2$ condition are found in the following corollary:

**Corollary 3.5** *The following statements are equivalent.*

*i)* $\Phi \in \Delta_2$

*ii)* *$E^\Phi(\mu)$ is weakly sequentially complete.*

*iii)* *$E^\Phi(\mu)$ has the Radon-Nikodym Property.*

*iv)* *$E^\Phi(\mu)$ is a dual Banach space.*

*Proof:* i)⇒ ii). If $\Phi \in \Delta_2$, then $E^\Phi(\mu) = L^\Phi(\mu)$ is weakly sequentially complete.
ii)⇒ i). If $E^\Phi(\mu)$ is weakly sequentially complete, then every closed subspace of $E^\Phi(\mu)$ is weakly sequentially complete. Consequently, $c_0$ cannot be isomorphic to a closed subspace of $E^\Phi(\mu)$ and so by corollary 3.3, $\Phi \in \Delta_2$.
i)⇒ iii). If $\Phi \in \Delta_2$, then $E^\Phi(\mu) = L^\Phi(\mu)$ has the Radon-Nikodym Property because $L^\Phi(\mu)$ is a separable dual space (see [4, Theorem III.3.1]).
iii)⇒ i). Suppose $E^\Phi(\mu)$ has the Radon-Nikodym Property. Hence, by [4, Theorem III.3.2], every closed subspace of $E^\Phi(\mu)$ has the Radon-Nikodym Property. Since $c_0$ lacks this property ([4, Example III.1.1]), $c_0$ cannot be a closed subspace of $E^\Phi(\mu)$, which means by corollary 3.3, that $\Phi \in \Delta_2$.
i)⇒ iv). If $\Phi \in \Delta_2$, then $E^\Phi(\mu) = L^\Phi(\mu)$ is well known to be a dual Banach space.
iv)⇒ i). Since $E^\Phi(\mu)$ is separable Banach space ([7, Theorem 3.5.1]), then if $E^\Phi(\mu)$ is a separable dual Banach space, by [4, Theorem III.3.1], $E^\Phi(\mu)$ has the Radon-Nikodym Property. In this form we have reduced our problem to the case iii)⇒ i). ∎

# 4 Some closing remarks

In [1] J. Alexopoulos has shown that if a bounded set with equi-absolutely continuous norms in any Orlicz space, is relatively weakly compact. Furthermore if $\Phi \in \Delta_2$ and its complement $\Psi$ satisfies $\lim_{t\to\infty} \frac{\Psi(ct)}{\Psi(t)} = \infty$ for some $c > 0$ then a bounded set $K \subset L_\Phi$ is relatively weakly compact if and only if $K$ has equi-absolutely continuous norms.

Making an effort to generalize this result, one can immediately ask several relevant questions:

1. One may ask whether given a $\Phi \in \Delta_2 \backslash \nabla_2$ with complementary function $\Psi$ satisfying $\liminf_{t\to\infty} \frac{\Psi(ct)}{\Psi(t)} < \infty$ for all $c > 0$, it is possible to find an $N$-function $F$ equivalent to $\Phi$ so that the complement $G$ of $F$ satisfies $\lim_{t\to\infty} \frac{G(ct)}{G(t)} = \infty$ for some $c > 0$.

   The answer to this question is negative for if there is a $\Phi \in \Delta_2 \setminus \nabla_2$ with its complement $\Psi$ satisfying $\liminf_{t\to\infty} \frac{\Psi(ct)}{\Psi(t)} < \infty$ for all $c > 0$ then if $F$ has complement $G$ and $F$ is equivalent to $\Phi$, then $G$ is equivalent to $\Psi$ and as such, there are positive constants $k_1$ and $k_2$ so that $\Psi(k_1 t) \leq G(t) \leq \Psi(k_2 t)$ for large values of $t$. Consequently for any $c > 0$ we have

$$\liminf_{t\to\infty} \frac{G(ct)}{G(t)} \leq \liminf_{t\to\infty} \frac{\Psi(k_2 ct)}{\Psi(k_1 t)} = \liminf_{s\to\infty} \frac{\Psi\left(\frac{k_2 c}{k_1} s\right)}{\Psi(s)} < \infty$$

2. Is weak compactness in $E^\Phi$ for $\Phi \notin \nabla_2$ equivalent to equi-absolute continuity of norms?

   One might think that by some density argument, theorem 2.3 holds for every weakly null sequence in such an $E^\Phi$. Nonetheless the answer to this question is negative for if in addition to $\Phi \notin \nabla_2$ we choose $\Phi \notin \Delta_2$ then corollary 3.4 guarantees the existence of a weakly null sequence in $E^\Phi$ that fails to have equi-absolutely continuous norms.

3. Related to corollary 3.3, we may ask wether $c_0$ itself maybe isomorphic to $E^\Phi(\mu)$. The answer to this question is NO, at least when $\mu$ is the Lebesgue measure on $[0, 1]$; since if $E^\Phi(\mu)$ is isomorphic to $c_0$, then $L^\Psi(\mu)$ is isomorphic to $\ell_1$ and so $L^\Psi(\mu)$ is separable. Therefore $\Psi \in \Delta_2$. This implies, by a theorem in [6], that $L^\Psi(\mu)$ lacks the Dunford Pettis Property contrary to the isomorphism between $L^\Psi(\mu)$ and $\ell_1$.

The following questions relating to the generalization of the theorem mentioned above, remain unresolved:

1. Does the complement $\Psi$ of every $\Phi \in \Delta_2 \setminus \nabla_2$ satisfy $\lim_{t\to\infty} \frac{\Psi(ct)}{\Psi(t)} = \infty$ for some $c > 0$?

2. Can the additional hypothesis of $\Phi \in \Delta_2$ be utilized to harvest the conclusion of theorem 2.3 for an arbitrary weakly null sequence $(f_n) \subset L^\Phi$?

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